Poisson Distr-n.

Recall: if $X_1, \dots, X_n$ are Poisson random variables with parameters $\lambda_1, \dots, \lambda_n$, then $X := \sum_{i=1}^{n} X_i$ is a Poisson random variable with parameter $\lambda = \lambda_1 + \dots + \lambda_n$.

Problem. Let $X(t)$ and $Y(t)$ be two independent Poisson processes with means λ and μ. Set $Z = X + Y$.

(a) Find $P(Z(3) = 4)$.

(b) Given that $X(1) = 1$, find $P(Z(1) = 3)$.

(c) Given that $Z(1) = 5$, find $P(Y(1) = 3)$.

Sol-n:

(a) $Z(t)$ is a Poisson process with param. $\lambda + \mu$, so

$$P(Z(3) = 4) = \frac{(3(\lambda+\mu))^4}{4!} e^{-3(\lambda+\mu)}$$

Rmk: we could come up with the same answer using that $4 = 4+0 = 3+1 = 2+2 = 1+3 = 0+4$ and, hence,

$$P(Z(3)=4) = P(X(3)=4) \cdot P(Y(3)=0) + P(X(3)=3) \cdot P(Y(3)=1) + \dots + P(X(3)=0) \cdot P(Y(3)=4) = 3^4 e^{-3\lambda} \cdot e^{-3\mu} \left(\frac{\lambda^4}{4!} + \frac{\lambda^3 \cdot \mu}{3! \cdot 1!} + \dots + \frac{\mu^4}{4!} \right) = \frac{3^4 (\lambda+\mu)^4}{4!} \cdot e^{-3(\lambda+\mu)}.$$

(b)
$$P(Z(1)=3 \mid X(1)=1) = \frac{P(Z(1)=3 \cap X(1)=1)}{P(X(1)=1)} = \frac{\cancel{P(X(1)=1)} \cdot P(Y(1)=2)}{\cancel{P(X(1)=1)}} =$$

$$= P(Y(1)=2) = \frac{\mu^2}{2!} e^{-\mu}.$$

(c)
$$P(Y(1)=3 \mid Z(1)=5) = \frac{P(Y(1)=3 \cap Z(1)=5)}{P(Z(1)=5)} =$$

$$= \frac{P(Y(1)=3) \cdot P(X(1)=2)}{P(Z(1)=5)} = \frac{\frac{\mu^3}{3!} \cdot \frac{\lambda^2}{2!} \cdot e^{-\mu} \cdot e^{-\lambda}}{\frac{(\lambda+\mu)^5}{5!} e^{-\lambda-\mu}} = \frac{5!}{2!\,3!} \cdot \frac{\lambda^2 \mu^3}{(\lambda+\mu)^5}$$

Problem 3, page 22.

Suppose that on average 10 people move into a city per week. Assume this is a Poisson process.

(a) Find the probability that 2 people move into the city the next day.

(b) Find the probability that the time until the next arrival is more than 2 days.

(c) Find the expected time until the 100^{th} arrival.

(d) Estimate the probability that 500^{th} arrival happens after more than one year.

Sol-n:

(a) $P(X(\frac{1}{7})=2)=\frac{\left(\frac{10}{7}\right)^2}{2!}e^{-\frac{10}{7}}=0.245.$

(b) Let T_n be the time between the $(n-1)^{st}$ and n^{th} arrivals. Then $P(T_1>t)=1-P(X(t)=0)=e^{-\lambda t}$ or $e^{-\frac{10}{7}t}$ in our case. This gives $P(T>2)=e^{-\frac{20}{7}}=0.06.$

(c) $T_1, T_2, \ldots, T_{100}$ are i.i.d. random variables with mean $\frac{1}{\lambda}=\frac{7}{10}$. Thus, $\mathbb{E}(T_1+T_2+\ldots+T_{100})=100\,\mathbb{E}(T_1)=70$ days or 70 weeks.

(d) Let $T = T_1 + \dots + T_{500}$, as T_i's are i.i.d. random variables and $500 \gg 0$ (is a large number), we apply the CLT:

$$\frac{T - n\mu}{\sqrt{n}\,\sigma(T_i)} = \frac{T - 500 \cdot 0.1}{\sqrt{500} \cdot 0.1} \sim N(0,1). \qquad \left(\mu = \frac{1}{\lambda} = \frac{1}{10}\right)$$

As there are 52 weeks in a year, we need

$$T > 52 \quad \text{or} \quad Z > \frac{52 - 500 \cdot 0.1}{\sqrt{500} \cdot 0.1} = 0.894$$

Finally, $P(Z > 0.894) = 1 - P(Z \leq 0.894) = 0.186$

Brownian Motion.

HW problem. Let $B(t)$ be a standard Brownian motion.

(a) Find $P(B(4)>1)$.

(b) Find $P(B(4)>1 \wedge B(7)-B(4)<2)$.

(c) Find $P(B(4)>3 \mid B(2)=1)$.

Sol-n:

(a) $P(\underset{\sim N(0,4)}{B(4)}>1) = P(Z>\frac{1}{2}) = 1-P(Z\leq\frac{1}{2}) = 1-0.691 = 0.309.$

(b) $P(B(4)>1 \wedge B(7)-B(4)<2) = P(B(4)>1)\cdot P(\underset{\sim N(0,3)}{B(7)-B(4)}<2) = 0.309\cdot P(Z<\frac{2}{\sqrt{3}}) = 0.309\cdot 0.876 = 0.271$

(c) $P(B(4)>3 \mid B(2)=1) =$

$= P(B(4)-B(2)>2 \mid B(2)=1) =$

$= P(\underset{\sim N(0,2)}{B(4)-B(2)}>2) = 1-P(Z\leq \sqrt{2}) \approx 0.079$

Use Ito's formula to compute the differentials:

1. $X(t, B(t)) = B^2(t)$.

$$dX(t,B(t)) = 2B(t)\,dB(t) + \frac{1}{2}\cdot 2\,dt = dt + 2B(t)\,dB(t)$$

2. $X(t, B(t)) = \ln(t + B^3(t))$

$$dX(t,B(t)) = \left(\frac{1}{t+B^3(t)} + \frac{6B(t)\cdot t + 6B^4(t) - 9B^4(t)}{(t+B^3(t))^2}\right)dt + \frac{3B^2(t)}{t+B^3(t)}\,dB(t)$$

$$\frac{\partial X}{\partial t} = \frac{1}{t+B^3(t)}$$

$$\frac{\partial X}{\partial B(t)} = \frac{3B^2(t)}{t+B^3(t)}$$

$$\frac{\partial^2 X}{\partial B(t)^2} = \frac{6B(t)(t+B^3(t)) - 9B^4(t)}{(t+B^3(t))^2}$$

Let $X(t)$ be the price of a stock at time t. If the current price is \$40 and we assume it can be modelled by a geom. Brownian motion with a drift parameter of 0.15 and volatility 0.6, find

(a) the probability that the price of the stock after 3 years is more than \$80.

(b) If the yearly interest rate is 3% and we want to sell an option to buy the stock for \$70 in 3 years, what should the price of the option be so there is no arbitrage opportunity?

Sol-n: Recall that geometric Brownian motion is given by $X(t) = C \cdot e^{\mu t + \sigma B(t)}$. From the data given, we have $X(t) = 40 e^{0.15t + 0.6 B(t)}$.

(a) $P(X(3) > 80) = P(40 e^{0.15 \cdot 3 + 0.6 \cdot B(3)} > 80) =$

$= P(e^{0.45 + 0.6 B(3)} > 2) = P(0.45 + 0.6 B(3) > \ln 2) =$

$= P\left(B(3) > \frac{\ln 2 - 0.45}{0.6}\right) \underset{\uparrow}{=} P\left(B(1) > \frac{\ln 2 - 0.45}{\sqrt{3} \cdot 0.6}\right) =$

$B(3) \sim N(0, 3)$

$= 1 - P(Z < 0.234) = 0.409$

The sol-n of the Black-Scholes eq-n (for the call option) corresponding to the geom. Brownian motion is

$$C(S,t) = S\cdot\varphi(\sigma\sqrt{T-t}+b) - Ee^{-r(T-t)}\varphi(b), \text{ where}$$

$$b = \frac{\ln(S/E)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

In our case (the initial time is $t=0$ and initial price of the stock $S=40$):

$$C(40,0) = 40\,\varphi(0.6\sqrt{3}-0.972) - 70e^{-0.09}\varphi(-0.972) =$$

$$= 40\,\varphi(0.0672) - 63.98\,\varphi(-0.972) = 40\cdot 0.527 -$$

$$-63.98\cdot 0.166 = \$10.459$$

$$\left(b = \frac{\ln(40/70)+(0.03-\frac{0.36}{2})\cdot 3}{0.6\sqrt{3}} = -0.972 \right).$$